The University of Texas at Austin
Dept. of Electrical and Computer Engineering
Midterm #1 ***Solution 6.0***

Date: October 3, 2024 Course: EE 313 Evans

Name: **Runner** **Blade**
Last, First

- **Exam duration**. The exam is scheduled to last 75 minutes.
- **Materials allowed**. You may use books, notes, your laptop/tablet, and a calculator.
- **Disable all networks**. Please disable all network connections on all computer systems. You may not access the Internet or other networks during the exam.
- **No AI tools allowed**. As mentioned on the course syllabus, you may not use GPT or other AI tools during the exam.
- **Electronics.** Power down phones. No headphones. Mute your computer systems.
- **Fully justify your answers**. When justifying your answers, reference your source and page number as well as quote the content in the source for your justification. You could reference homework solutions, test solutions, etc.
- **Matlab**. No question on the test requires you to write or interpret Matlab code. If you base an answer on Matlab code, then please provide the code as part of the justification.
- **Put all work on the test**. All work should be performed on the quiz itself. If more space is needed, then use the backs of the pages.
- **Academic integrity.** By submitting this exam, you affirm that you have not received help directly or indirectly on this test from another human except the proctor for the test, and that you did not provide help, directly or indirectly, to another student taking this exam.

	Problem	*Point Value*	*Your score*	*Topic*
Rick	1	24		Sinusoidal Signals
Roy	2	24		Fourier Series
Rachel	3	26		Sampling
Pris	4	24		Potpourri
	Total	100		

Please note that the total is 98 points. Two points were added to everyone's test

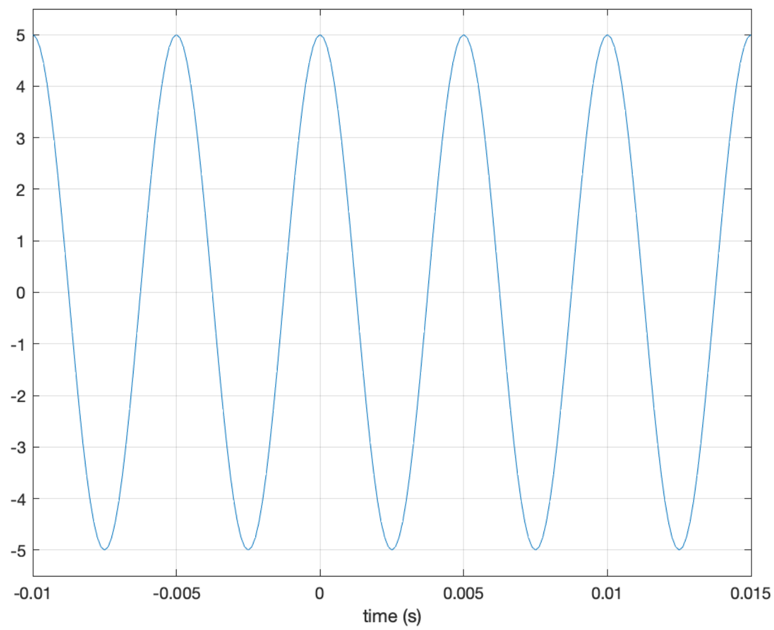
SPFirst Sec. 2-1 to 2-3	*Lecture Slide 1-16*
	Homework Prob. 1.1

Problem 1.1 *Sinusoidal Signals*. 24 points.

Consider the sinusoidal signal $x(t) = A \cos(2\,\pi\, f_0\, t\ +\ \theta)$ with

- amplitude A
- continuous-time frequency f_0 in Hz
- phase θ in radians

From the plot of $x(t)$ on the right,



(a) Estimate the amplitude A. Explain how you estimated the value of this parameter. *6 points.*

In a sinusoidal signal, the peak occurs at $\boldsymbol{A}$ and valley occurs at $\boldsymbol{-A}$ because the values of cosine are in the interval $\boldsymbol{[-1, 1]}$. From the plot, $\boldsymbol{A = 5}$.

(b) Estimate the continuous-time frequency f_0 in Hz. Explain how you estimated the value of this parameter. *6 points.*

***Approach #1*: The duration from the peak at time $\boldsymbol{t = 0}$ and the next peak at $\boldsymbol{t = 0.005s}$, which is $\boldsymbol{0.005s}$, represents the fundamental period $\boldsymbol{T_0}$ of the sinusoidal signal. The fundamental frequency $\boldsymbol{f_0 = \frac{1}{T_0} = \frac{1}{0.005s} = 200\ \text{Hz}}$.**

***Approach #2*: From counting the peaks, the plot contains exactly five periods and lasts for $\boldsymbol{0.025s}$. Hence, the fundamental period is $\boldsymbol{T_0 = 0.005s}$ and the fundamental frequency is $\boldsymbol{f_0 = \frac{1}{T_0} = \frac{1}{0.005s} = 200\ \text{Hz}}$.**

***Approach #3*: The plot contains 10 zero crossings and lasts for $\boldsymbol{0.025s}$. Each fundamental period has two zero crossings; hence, there are five periods in $\boldsymbol{0.025s}$. The fundamental period is $\boldsymbol{T_0 = 0.005s}$ and fundamental frequency is $\boldsymbol{f_0 = \frac{1}{T_0} = \frac{1}{0.005s} = 200\ \text{Hz}}$.**

(c) Estimate the phase θ in radians. Explain how you estimated the value of this parameter. *6 points*.

A peak value of 5 occurs at $\boldsymbol{t = 0}$. From the answer in part (a), $\boldsymbol{A = 5}$.

Hence, $\boldsymbol{x(0) = 5\cos(\theta) = 5}$, which means $\boldsymbol{\theta = 0}$ rad.

(d) What is the phase of the signal $x(t - 0.001s)$? Please show your intermediate steps. *6 points*.

$$\boldsymbol{x(t-0.001) = A\cos(2\,\pi\, f_0\,(t-0.001s)\ +\ \theta) = A\cos(2\,\pi\, f_0\, t - 2\,\pi\, f_0\,(0.001) + \theta)}$$

Hence, the phase is $\boldsymbol{-2\,\pi\, f_0\,(0.001s) + \theta = -2\,\pi\,(200\ Hz)(0.001s) + 0 = -0.4\,\pi}$.

SPFirst Sec. 2-3.2	*Lecture Slide 2-4*

Problem 1.2. *Fourier Series Properties.* 24 points.

SPFirst Sec. 3-3 to 3-5 | *Lecture slides 3-7 to 3-14* | *Homework Prob. 2.4 & 3.1*

The continuous-time Fourier series has several properties.

For example, if $y(t) = A\, x(t)$ and $x(t)$ is periodic with fundamental frequency f_0 and Fourier series coefficients a_k, then the Fourier series coefficients b_k for $y(t)$ can be found using $b_k = A\, a_k$:

$$y(t) = A\, x(t) = A \sum_{k=-\infty}^{\infty} a_k\, e^{j2\pi(kf_0)t} = \sum_{k=-\infty}^{\infty} A\, a_k\, e^{j2\pi(kf_0)t}$$

For the following expressions, derive the relationship between the Fourier series coefficients b_k for $y(t)$ and the Fourier series coefficients a_k for $x(t)$ where

$$a_k = \frac{1}{T_0}\int_0^{T_0} x(t)\, e^{-jk\omega_0 t}\, dt$$

(a) $y(t) = x(t) + C$ where C is a constant. *6 points.*

Adding a constant C should only affect the DC term of the Fourier series:

$$b_0 = \frac{1}{T_0}\int_0^{T_0} (x(t) + C)\, dt = \frac{1}{T_0}\int_0^{T_0} x(t)\, dt + \frac{1}{T_0}\int_0^{T_0} C\, dt = a_0 + C$$

For $k \neq 0$,

$$b_k = \frac{1}{T_0}\int_0^{T_0} (x(t) + C)\, e^{-jk\omega_0 t}\, dt = \underbrace{\frac{1}{T_0}\int_0^{T_0} x(t)\, e^{-jk\omega_0 t}\, dt}_{a_k} + \frac{1}{T_0}\int_0^{T_0} C\, e^{-jk\omega_0 t}\, dt$$

Working on the last term:

SPFirst Sec. 3-4.2

$$\frac{1}{T_0}\int_0^{T_0} C\, e^{-jk\omega_0 t}\, dt = C\frac{1}{T_0}\left[\frac{e^{-jk\omega_0 t}}{-jk\omega_0}\right]_0^{T_0} = C\frac{1}{T_0}\left(\frac{e^{-jk\omega_0 T_0}}{-jk\omega_0} + \frac{1}{jk\omega_0}\right) = C\frac{1}{T_0}\left(-\frac{e^{-j2\pi k}}{jk\omega_0} + \frac{1}{jk\omega_0}\right) = 0$$

because $e^{-j2\pi k} = \cos(-2\pi k) = 1$. So, $b_0 = a_0 + C$ and $b_k = a_k$ for $k \neq 0$.

(b) $y(t) = \sin(2\,\pi\, f_0\, t)\, x(t)$. This is a type of amplitude modulation. *9 points.*

SPFirst Sec. 3-2 | *Fall 2021 Midterm 1.4(c)*

Hint: If $y(t) = \cos(2\,\pi\, f_0\, t)\, x(t)$, then $b_k = \frac{1}{2}\, a_{k-1} + \frac{1}{2}\, a_{k+1}$.

From Fall 2021 Midterm 1.4(c), multiplying a periodic signal $x(t)$ by $\cos(2\,\pi\, f_0\, t)$ causes two effects in the resulting signal $y(t)$ per $b_k = \frac{1}{2}\, a_{k-1} + \frac{1}{2}\, a_{k+1}$. The kth harmonic frequency in $y(t)$ is the $(k-1)$th harmonic frequency in $x(t)$ shifted to the right by f_0 and scaled by ½ plus the $(k+1)$th harmonic frequency in $x(t)$ shifted to the left by f_0 and scaled by ½.

***Approach #1*. From Fall 2021 Midterm 1.4(c), the expression $b_k = \frac{1}{2}\, a_{k-1} + \frac{1}{2}\, a_{k+1}$ comes from the inverse Euler's formula $\cos(2\,\pi\, f_0\, t) = \frac{1}{2} e^{-j\,2\,\pi\, f_0\, t} + \frac{1}{2} e^{j\,2\,\pi\, f_0\, t}$. A similar amplitude modulation effect occurs when multiplying a signal $x(t)$ by $\sin(2\,\pi\, f_0\, t)$. The inverse Euler's formula is $\sin(2\,\pi\, f_0\, t) = -\frac{1}{2j} e^{-j\,2\,\pi\, f_0\, t} + \frac{1}{2j} e^{j\,2\,\pi\, f_0\, t}$. By interference,**

$$b_k = -\frac{1}{2j}\, a_{k-1} + \frac{1}{2j}\, a_{k+1} = \frac{j}{2} a_{k-1} - \frac{j}{2}\, a_{k+1} \qquad (\textit{has sign error})$$

The correct answer is

$$b_k = -\frac{j}{2} a_{k-1} + \frac{j}{2}\, a_{k+1}$$

***Approach #2*:**

$$b_k = \frac{1}{T_0}\int_0^{T_0} y(t)\, e^{-j2\pi(kf_0)t}\, dt = \frac{1}{T_0}\int_0^{T_0} \sin(2\,\pi\, f_0\, t)\, x(t)\, e^{-j2\pi(kf_0)t}\, dt$$

$$b_k = \frac{1}{T_0}\int_0^{T_0} \left(-\frac{1}{2j}\, e^{-j2\pi f_0 t} + \frac{1}{2j}\, e^{j2\pi f_0 t}\right) x(t)\, e^{-j2\pi(kf_0)t}\, dt$$

$$b_k = -\frac{1}{2j}\left(\frac{1}{T_0}\int_0^{T_0} x(t)\, e^{-j2\pi f_0 t}\, e^{-j2\pi(kf_0)t}\, dt\right) + \frac{1}{2j}\left(\frac{1}{T_0}\int_0^{T_0} x(t)\, e^{j2\pi f_0 t}\, e^{-j2\pi(kf_0)t}\, dt\right)$$

$$b_k = -\frac{1}{2j}\left(\frac{1}{T_0}\int_0^{T_0} x(t)\, e^{-j2\pi\left((k+1)f_0\right)t}\, dt\right) + \frac{1}{2j}\left(\frac{1}{T_0}\int_0^{T_0} x(t)\, e^{-j2\pi\left((k-1)f_0\right)t}\, dt\right)$$

$$b_k = -\frac{1}{2j}\, a_{k+1} + \frac{1}{2j}\, a_{k-1} = \frac{j}{2}\, a_{k+1} - \frac{j}{2} a_{k-1} = -\frac{j}{2} a_{k-1} + \frac{j}{2}\, a_{k+1}$$

Multiplying a periodic signal $x(t)$ by $\sin(2\,\pi\, f_0\, t)$ causes two effects in the resulting signal $y(t)$ per $b_k = -\frac{j}{2}\, a_{k-1} + \frac{j}{2}\, a_{k+1}$. The kth harmonic frequency in $y(t)$ is the $(k-1)$th harmonic frequency in $x(t)$ shifted to the right by f_0 and scaled by $-\frac{j}{2}$ plus the $(k+1)$th harmonic frequency in $x(t)$ shifted to the left by f_0 and scaled by $\frac{j}{2}$.

Solution given in SPFirst Problem P3-14 on page 67

(c) $y(t) = \frac{d}{dt}\, x(t)$. *9 points.*

$$y(t) = \frac{d}{dt}\, x(t) = \frac{d}{dt}\left(\sum_{k=-\infty}^{\infty} a_k\, e^{j2\pi(kf_0)t}\right) = \sum_{k=-\infty}^{\infty} a_k\, \frac{d}{dt}\left(\, e^{j2\pi(kf_0)t}\right)$$

$$y(t) = \sum_{k=-\infty}^{\infty} \left(j2\pi(kf_0)\right)\; a_k\;\; e^{j2\pi(kf_0)t}$$

Hence,

$$b_k = j\, 2\pi\, k\, f_0\, a_k$$

Fall 2023 Midterm 1.3 | Fall 2018 Final Exam Prob 8(c) | Homework Prob. 2.1 & 3.1

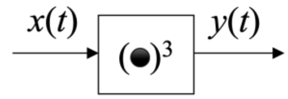
In-Lecture Assignments 1 & 2 | SPFirst Sec. 1-2 & 1-3 | Lecture Slide 1-20

SPFirst Sec. 2-2 & 2-4 | Lecture Slides 2-5 & 2-6

Problem 1.3. *Sampling*. 26 points**.**

(a) Let $x(t) = \cos(2\pi f_0 t)$ be a continuous-time signal for $-\infty < t < \infty$.

i. From the block diagram on the right, $y(t) = x^3(t)$.



Write $y(t)$ as a sum of cosines. *6 points*.

$\boldsymbol{x(t)}$ **has frequencies** $\boldsymbol{-f_0}$ **and** $\boldsymbol{+f_0}$ **because** $\boldsymbol{x(t) = \cos(2\pi f_0 t) = \frac{1}{2} e^{-j2\pi f_0 t} + \frac{1}{2} e^{j2\pi f_0 t}}$

***Approach #1*: From *Signal Processing First* Section 3-5 and Lecture Slide 3-9,**

$$y(t) = x^3(t) = \cos^3(2\pi f_0 t) = \frac{3}{4}\cos(2\pi f_0 t) + \frac{1}{4}\cos(2\pi(3f_0)t)$$

***Approach #2*:**

$$y(t) = x^3(t) = \cos^3(2\pi f_0 t) = \cos^2(2\pi f_0 t)\cos(2\pi f_0 t)$$

$$y(t) = \left(\frac{1}{2} + \frac{1}{2}\cos(2\pi(2f_0)t)\right)\cos(2\pi f_0 t)$$

$$y(t) = \frac{1}{2}\cos(2\pi f_0 t) + \frac{1}{2}\cos(2\pi(2f_0)t)\cos(2\pi f_0 t)$$

The second term gives beat frequencies of $\boldsymbol{f_0}$ **and** $\boldsymbol{3f_0}$ **:**

$$y(t) = \frac{1}{2}\cos(2\pi f_0 t) + \frac{1}{4}\cos(2\pi f_0 t) + \frac{1}{4}\cos(2\pi(3f_0)t)$$

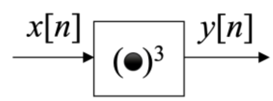
$$y(t) = \frac{3}{4}\cos(2\pi f_0 t) + \frac{1}{4}\cos(2\pi(3f_0)t)$$

ii. Let $f_0 = 3000\ Hz$. What negative, zero, and positive frequencies are present in $y(t)$? *6 points*

$\boldsymbol{-3f_0, -f_0, f_0,}$ **and** $\boldsymbol{3f_0}$ **which are** $\boldsymbol{-9000}$ **Hz,** $\boldsymbol{-3000}$ **Hz, 3000 Hz, and 9000 Hz**

(b) Let $x(t) = \cos(2\pi f_0 t)$ be a continuous-time signal for $-\infty < t < \infty$. Discrete-time signal $x[n]$ is obtained by sampling $x(t)$, and $y[n]$ is obtained by sampling $y(t)$.

i. From the block diagram below, $y[n] = x^3[n]$.



Write it as a sum of cosines. *6 points*

Discrete-time signal $\boldsymbol{x[n]}$ **is obtained by sampling** $\boldsymbol{x(t)}$**:**

$$x[n] = x(t)|_{t=nT_s} = \cos\big(2\pi f_0 (nT_s)\big) = \cos(2\pi f_0 T_s n) = \cos\left(2\pi \frac{f_0}{f_s} n\right)$$

$\boldsymbol{x[n] = \cos(\hat{\omega}_0 n)}$ **where** $\boldsymbol{\hat{\omega}_0 = 2\pi \frac{f_0}{f_s}}$**.**

From *Signal Processing First* Section 3-5 and Lecture Slide 3-9,

$$y[n] = x^3[n] = \cos^3(\hat{\omega}_0 n) = \frac{3}{4}\cos(\hat{\omega}_0\, n) + \frac{1}{4}\cos(3\,\hat{\omega}_0\, n)$$

ii. Let $f_0 = 3000$ Hz and $f_s = 8000$ Hz. What negative, zero and positive discrete-time frequencies are present in $y[n]$ between $-\pi$ rad/sample and π rad/sample? What are their corresponding continuous-time frequencies? *8 points*.

***Approach #1*: By sampling** $\boldsymbol{y(t)}$ **to obtain** $\boldsymbol{y[n]}$ **using sampling rate** $\boldsymbol{f_s = 8000}$ **Hz, the frequency component of 9000 Hz will alias to 9000 Hz – 8000 Hz = 1000 Hz and the**

frequency component at -9000 Hz will alias to -9000 Hz + 8000 Hz = -1000 Hz. The discrete-time frequencies corresponding to -3000, -1000, 1000, and 3000 Hz with respect to $f_s = 8000$ Hz are $-2\pi\frac{3}{8}$, $-2\pi\frac{1}{8}$, $2\pi\frac{1}{8}$, and $2\pi\frac{3}{8}$ rad/sample.

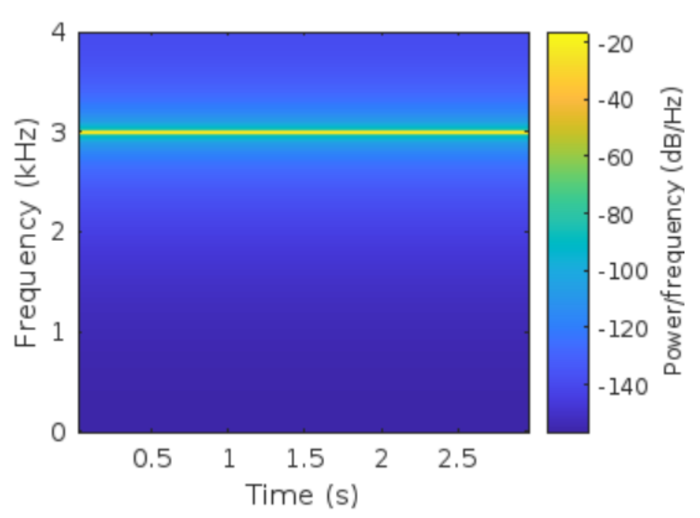
***Approach #2*: Here, $\hat{\omega}_0 = 2\pi\frac{f_0}{f_s} = 2\pi\frac{3000\ Hz}{8000\ Hz} = 2\pi\frac{3}{8}$ rad/sample. The other positive frequency in $y[n]$ is $3\ \hat{\omega}_0 = 3\left(2\pi\frac{3}{8}\right) = 2\pi\frac{9}{8}$. This aliases to $2\pi\frac{9}{8} - 2\pi = 2\pi\frac{1}{8}$ rad/sample and corresponds to continuous-time frequency $f_1 = \frac{\hat{\omega}_0}{2\pi}\ f_s = 1000$ Hz. The discrete-time frequencies are $-2\pi\frac{3}{8}$, $-2\pi\frac{1}{8}$, $2\pi\frac{1}{8}$, and $2\pi\frac{3}{8}$ rad/sample which correspond to -3000, -1000, 1000, and 3000 Hz when using $f_s = 8000$ Hz .**

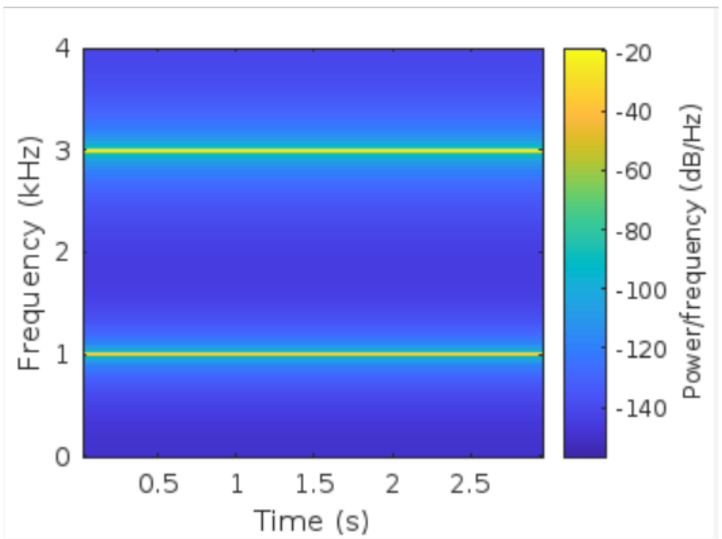
We can use Matlab for part (b) by modifying the Matlab code in Fall 2023 Midterm 1 Problem 2, which had two sinusoidal signals being input into a squaring block instead of one sinusoidal signal being input into a cubing block:

```
fs = 8000;
Ts = 1/ fs;
tmax = 3;
t = 0 : Ts : tmax;
f0 = 3000;

x = cos(2*pi*f0*t);
figure;
spectrogram(x, 512, 256, 512, fs, 'yaxis');

y = x.^3;
figure;
spectrogram(y, 512, 256, 512, fs, 'yaxis');
```





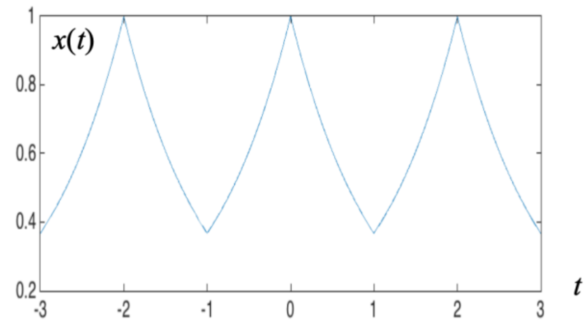
Problem 1.4. *Potpourri*. 24 points.

(a) Consider the periodic signal $x(t)$ on the right with a fundamental period of $T_0 = 2$ seconds.

Over one fundamental period,

$$e^{t} \quad \text{for } -\frac{T_0}{2} \le t < 0$$

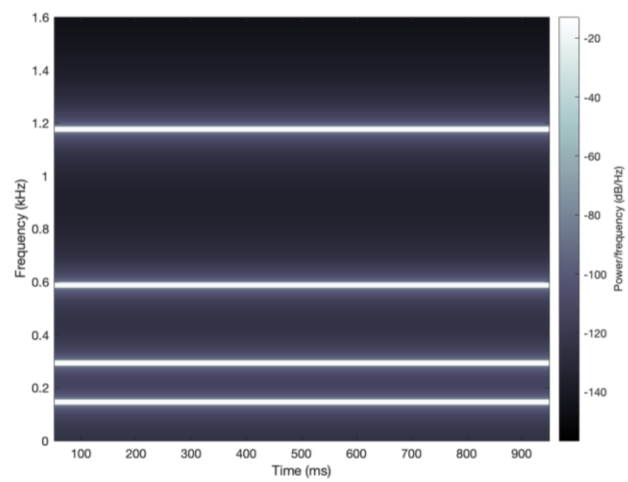
$$e^{-t} \quad \text{for } 0 \le t < \frac{T_0}{2}$$



If we keep a large but finite number of Fourier series coefficients, explain whether or not the Fourier synthesis will suffer from Gibbs phenomenon. In your answer, please explain what Gibbs phenomenon is. *12 points*.

No, $x(t)$ will not suffer from Gibbs' phenomenon because it does not have any amplitude discontinuities (*SPFirst Sec. 3-6.6*). Gibbs' phenomenon occurs at/near each amplitude discontinuity when using a finite number of Fourier series synthesis terms. An amplitude discontinuity could occur at or inside the boundaries of the fundamental period, depending what interval of time is chosen for the fundamental period. No matter how many finite terms are used for Fourier series synthesis, oscillation will be seen at both amplitude values across the amplitude discontinuity. This oscillation will lead to a worst-case error between the Fourier series synthesis and the original signal on either side of the amplitude discontinuity of no less than ±9% regardless of how many finite terms are used (*SPFirst Sec. 3-6.6*). Fig. 3-17 on page 54 of *SPFirst* **shows Gibbs' phenomenon for the square wave. Gibbs' phenomenon does not occur for a periodic signal that does not have an amplitude discontinuity, even if the periodic signal has a discontinuity in derivative at one of more points. Gibbs' phenomenon does not exist if an infinite number of terms is used.**

(b) Consider the spectrogram of a signal $y(t)$ given below. *12 points*.



i. How would you describe the relationship among the frequencies in $y(t)$.

The signal consists of four principal frequencies of approximate values

$\boldsymbol{f_1 =}$ **150 Hz**
$\boldsymbol{f_2 =}$ **300 Hz**
$\boldsymbol{f_3 =}$ **600 Hz**
$\boldsymbol{f_4 =}$ **1200 Hz**

The frequencies are octave-spaced; i.e., $\boldsymbol{f_2}$ is twice $\boldsymbol{f_1}$, and $\boldsymbol{f_3}$ is twice $\boldsymbol{f_2}$ and so forth.

ii. Please give an equation for $y(t)$ over the time plotted $0 \le t \le 1$.

Answer #1: **All four principal frequencies are equal in strength because they have the same intensity along all four horizontal lines. We'll call the amplitude $\boldsymbol{A}$. Spectrograms only show the magnitude of the frequency components; the phases are unknown:**

$$y(t) = A\cos(2\pi f_1 t + \theta_1) + A\cos(2\pi f_2 t + \theta_2) + A\cos(2\pi f_3 t + \theta_3) + A\cos(2\pi f_4 t + \theta_4)$$

SPFirst Sec. 3-7 & 3-8 | *Lecture Slides 4-4 & 4-12* | *Homework Prob. 2.3 & 2.4*

Mini-Project #1 | *Midterm 1.4 for F17, F18, F21 & F23*

Answer #2: **The upper two frequencies at 600 Hz and 1200 Hz are beat frequencies of 900 Hz and 300 Hz, and the lower two frequencies of 150 Hz and 300 Hz are beat frequencies of 225 Hz and 75 Hz:**

$$\boldsymbol{y(t) = C\cos(2\pi(900\text{Hz})t + \theta_A)\cos(2\pi(300\text{Hz})t + \theta_B) +}$$
$$\boldsymbol{C\cos(2\pi(225\text{Hz})\,t + \theta_C)\cos(2\pi(75\text{Hz})t + \theta_D)}$$

% Matlab code for the plot for Problem 1.1

```
phi = 0;
f0 = 200;
T0 = 1/f0;
fs = 40*f0;
Ts = 1/fs;
tmin = -2/f0;
tmax = 3/f0;
t = tmin : Ts : tmax;
A = 5;
x = A*cos(2*pi*f0*t + phi);
figure;
plot(t, x);
xlim( [tmin tmax] );
ylim( [-1.1*A 1.1*A] );
xlabel('time (s)');
grid on;
```

% Matlab code for the plot of the periodic signal in Problem 1.4(a)

```
T0 = 2;
f0 = 1/T0;
fs = 400*f0;
Ts = 1/fs;

% Generate three periods
t = -(3/2)*T0 : Ts : (3/2)*T0;
tmod = mod((t - T0/2), T0) - T0/2;
x = exp(-abs(tmod));

figure;
plot(t, x);
ylim( [0.2, 1.2] );
xlabel( 't' );
ylabel( 'x(t)' );
```

% Matlab code for the spectrogram plot in Problem 1.4(b).

```
fs = 11025;
Ts = 1/fs;
t = 0 : Ts : 1;
fD3 = 147;       % Musical note D, third octave, Western scale
x1 = cos(2*pi*fD3*t);
x2 = cos(2*pi*(2*fD3)*t);
x3 = cos(2*pi*(4*fD3)*t);
x4 = cos(2*pi*(8*fD3)*t);
x = x1 + x2 + x3 + x4;
N = 1575;
overlap = round((5/7)*N);
figure;
spectrogram(x, N, overlap, N, fs, 'yaxis');
ylim( [0 1.6] );
colormap('bone');
```